Polarities and Trialities in Geometry

Let $X = (P, L, I)$ be a finite connected incidence system such that each element of P (resp. ly, of L) is incident with $s + 1$ elements of L (resp. ly, of P), $s \geq 2$. Elements of P are called *points* of X and those of L are called *lines* of X.

1 Polarities

A

$$polarity of$$

X$is, an involutory, incidence preserving, bijection \tau$ of $P \cup L$ which interchanges P and L.

Clearly, a polarity is unique, up to an isomorphism of the incidence system X. Recall that an *automorphism* of X is a pair (φ, ψ) where φ is a bijection of P and ψ is a bijection of L such that, for $(x, y) \in P \times L$, xIy if, and only if, $\varphi(x) I \psi(y)$.

If $s = 1$, X is a regular graph of degree 2 with point set P and line set L . (Its connected components are cycles on even number of verticies.) NOT NECESSARILY A PARTIAL GEOMETRY.

Existence of τ implies that the number of elements of L containing two distinct elements x and y of P is equal to the number of elements of P incident with x^τ and y^τ of L.

Let $\mathcal{O}$ (resp. ly $\mathcal{S}$) be the set of all absolute points (resp. ly absolute lines) of τ. That is, $\mathcal{O} = \{x \in P : x \in x^\tau\}$ and $\mathcal{S} = \{l \in L : l^\tau \in l\}$.Give example when $\mathcal{O}$,$\mathcal{S}$ or both can be empty.

1.1 Polarities of Generalized polygons

1.1.1 Polarities of Projective planes

1.1.2 polarities of generalized quadrangles

1.1.3 polarities of generalized hexagons

1.2 Polarities of metasymplectic spaces (Geometries of type F_4)

1.3 Polarities of geometries of type E_6

$\tau\mathcal{O}\mathcal{S}\mathcal{O}\mathcal{S}PL[$ `If` $Xp \in \mathcal{O}p^\tau$ $\mathcal{S}p.Then, is a bijection between and. Note that (so, also) could be empty or equal to (so,).$

`We say that` O `is an ovoid in` X `if each maximal subspace of` X `which is isotropic with respect to` τ `contains a unique element of` O. `We say that` S `is a spread in` X `if ?? Are the set` $\mathcal{O}$OF ISOTROPIC POINTS AND THE SET L OF ISOTROPIC LINES OF A POLARITY ALWAYS OVOIDS, SPREADS IN THIS SENSE? *Study the polarity and the bilinear maps and the polarity of the code associated with subspaces in the geometry defined by the lines of an* *incidence system with a given polarity.* If X is a partial linear space, then for $p \in \mathcal{O}$, p^τ is the only element of $\mathcal{S}$ incident with p; and dually. Examples of polarities of interest are of :

1. The incidence system $\mathcal{P}^r_{k,n-k-1}(n,q) = (\mathcal{P}_k(n,q), \mathcal{P}_{n-k-1}(n,q), I_r)$,$0 \leq k \leq [n/2]-1$ and $-1 \leq r \leq k$, where $\mathcal{P}_s(n,q)$ is the set of $s-$flats in the the projective space $\mathcal{P}(n,q)$ of dimension n over $\mathbb{F}_q$ with a $k-$flat A defined to be I_r- incident with a $(n-k-1)-$flat B if $A \cap B$ is an $r-$flat. Any polarity of $\mathcal{P}(n,q)$ induces a polarity on each $\mathcal{P}^r_{k,n-k-1}(n,q)$. For a given r, does there exist any other polarities? What is the relation between the polarities of $\mathcal{P}(n,q)$ and those of $P^r_{k,n-k-1}(n,q)$.

2. The other finite lie incidence structures;

3. finite projective planes;

4. finite symmetric designs;

5. the generalized quadrangle $W(2^{2n+1})$, $n \geq 1$;

6. the generalized hexagon $H(3^{2n+1})$, $n \geq 1$;

7. the metasymplectic spaces;

8. Any interesting examples in dual polar spaces, parapolar spaces, lie incidence systems,

Remark 1 *A feature of interest in Example 5 (resp. ly Example 6; the case $n = 8$ and $p = 2$ of Example 1) is that $\mathcal{O}$ is an ovoid of $W(2^{2n+1})$ (resp. ly of $H(3^{2n+1})$; ?? of $\Omega(4,2^e)$ -geometry) and that the characteristic functions of the tangent planes to this ovoid is a basis for the p$-$ ary code generated by the hyperplanes of the incidence system. Also, there is a triality associated with the $\Omega(4,2^e)$ -geometry. What is its relation to the incidence system?*

1.3.1

Polarities of a symmetric $(\mathbf{v},\mathbf{k},\lambda)$- designs $(\mathbf{P},\mathbb{B})$

For distinct points x,y, the *line xy* is the intersection of all blocks containing x and y. Two points are on exactly one line and a line of size (v$\boxed{2212}$ $\boxed{3BB}$)/(k$\boxed{2212}$ $\boxed{3BB}$) has nonempty intersection with each block [Dembowski and Wagner, Arch. Math. 11 (1960) 465-469]. Assume that D admits a null polarity $\boxed{3C4}$; that is, x$\boxed{2208}$ x$\boxed{3C4}$ for each point x. We say that a line is singular if it contains distinct points x,y with y$\boxed{2208}$ x$\boxed{3C4}$; and nonsingular otherwise. If x$\boxed{2062}$ y is a singular line and x$\boxed{2260}$ z$\boxed{2208}$ x$\boxed{2062}$ y, then z$\boxed{2208}$ x$\boxed{27C2}$ $\boxed{2229}$ y$\boxed{27C2}$, and so, x,z$\boxed{2286}$ z$\boxed{27C2}$, y$\boxed{2208}$ x$\boxed{2062}$ y=x$\boxed{2062}$ z$\boxed{2286}$ z$\boxed{27C2}$. The equality x$\boxed{2062}$ y=x$\boxed{2062}$ z holds because: every block containing x and y contains z also and so, the blocks containing {x,y} and {x,z} are the same. Any line in D is either singular or nonsingular. Examples: (1) Projective geometries P$\boxed{2062}$ G$\boxed{2061}$ (d,q)(2) Highman's orthogonal symmetric designs having the same parameters as in (1): Points are the singular points of a d-dimensional orthogonal Fq-space with both d and q odd; its blocks correspond to the hyperplanes x$\boxed{27C2}$. 1. (Dembowski-Wagner) The projective space is the only symmetric design such that all lines have size (v$\boxed{2212}$ $\boxed{3BB}$)/(k$\boxed{2212}$ $\boxed{3BB}$). 2. (Kantor) Let D be a symmetric design admitting null polarity. (a) If all singular lines have size (v$\boxed{2212}$ $\boxed{3BB}$)/(k$\boxed{2212}$ $\boxed{3BB}$)>$\boxed{3BB}$, then D is either a projective space or an orthogonal design. (b) If all nonsingular lines have size (v$\boxed{2212}$ $\boxed{3BB}$)/(k$\boxed{2212}$ $\boxed{3BB}$), then D is a projective space. Survey all known examples1. Which projective planes admit polarities? Desarguesian planes do and the polarities are either orthogonal or unitary. 2. For Desarguesian spaces, polarities exist and they are either orthogonal, alternating or unitary. 3. For a polarity of a Desarguesian projective plane, the set of absolute points is nonempty. It has q+1 points if the polarity is orthogonal and s3+1 points if the polarity is unitary and q=s2. S$\boxed{2062}$ t$\boxed{2062}$ a$\boxed{2062}$ bP$\boxed{2062}$ G$\boxed{2062}$ L3$\boxed{2061}$ (q)$\boxed{2061}$ (Conic)=P$\boxed{2062}$ G$\boxed{2062}$ L2$\boxed{2061}$ (q)={y$\boxed{21A6}$ (a$\boxed{2062}$ y+b)/c$\boxed{2062}$ y+d:a$\boxed{2062}$ d$\boxed{2212}$ b$\boxed{2062}$ c$\boxed{2260}$ 0}.

Theorem 2 *Let $\boxed{393}$ be a projective plane of order q and $\boxed{3B8}$ be a polarity of $\boxed{393}$. Then, the number N$\boxed{3B8}$ of absolute points with respect to $\boxed{3B8}$ satisfy q+1$\boxed{2264}$ N$\boxed{3B8}$ $\boxed{2264}$ q$\boxed{2062}$ q+1. (i) If N$\boxed{3B8}$ =q+1, the set of absolute points is an oval if q is odd and the absolute points is collinear if q is even. (ii) If N$\boxed{3B8}$ =*

+q $\sqrt{q}$, the set of absolute points, with sets of points incident with secant lines, form a unital.

Also, if q is not a square, then $N_\theta = q + 1$.

Proof. ... ■

Definition 3 *A unital of order q is a set of $q^3 + 1$ points having $q^2\left(q^3+1\right)/2$ distinguished subsets containing $q + 1$ elements each, called blocks, such that each block is determined by any 2 of its points.*

Question: Determine the structure of the subcode of $\mathbb{F}_q$-code of lines of a projective plane π of order q generated by conics of π if q odd and hyperovals of π if q is even.

Problem 4 *For $l \in \mathcal{L}$, can we say anything in general about $|l \cap \mathcal{O}|$?*

1. *Let $X = (P, L, I)$ be a finite connected incidence system such that each element of P (resp. ly, of L) is incident with $s + 1$ elements of L (resp. ly, of P), $s \geq 2$. If $s = 1$, X is a regular graph of degree 2 with point set P and line set L . (Its connected components are cycles on even number of verticies.)* Not

NECESSARILY A PARTIAL GEOMETRY *Assume that* X *admits a polarity; that is, an involutive, incidence preserving, bijection* τ *of* $P \cup L$ *which interchanges* P *and* L. *Clearly, a polarity is unique up to an isomorphism of the incidence system* X. *Existence of* τ *implies that the number of elements of* L *containing two distinct points* x *and* y *is equal to the number of elements of* P *incident with* x^{τ} *and* y^{τ} *of* L. *Recall that an automorphism of* X *is a pair* (φ, ψ) *where* φ *is a bijection of* P *and* ψ *is a bijection of* L *such that, for* $(x, y) \in P \times L$, xIy *if, and only if,* $\varphi(x) I \psi(y)$. *Let* $\mathcal{O}$ *(resp. ly* $\mathcal{S}$*) be the set of all absolute points (resp. ly absolute lines) of* τ. *That is,* $\mathcal{O} = \{x \in P : x \in x^{\tau}\}$ *and* $\mathcal{S} = \{l \in L : l^{\tau} \in l\}$. *Then,* τ *is a bijection between* $\mathcal{O}$ *and* $\mathcal{S}$. *Note that* $\mathcal{O}$ *(so,* $\mathcal{S}$ *also) could be empty or equal to* P *(so,* L*).*[GIVE EXAMPLES WHEN O (SO, S ALSO) ARE EMPTY.] *If* X *is a partial linear space, then, for* $p \in \mathcal{O}$, p^{τ} *is the only element of* $\mathcal{S}$ *incident with* p, *and dually.*

We say that O *is an ovoid in* X *if each maximal subspace of* X *which is isotropic with respect to* τ *contains a unique element of* O. *We say that* S *is a spread in* X *if ?? Are the set* $\mathcal{O}$OF ISOTROPIC POINTS AND THE SET L OF ISOTROPIC LINES OF A POLARITY ALWAYS OVOIDS, SPREADS IN THIS SENSE? *Study the polarity and the bilinear maps and the polarity of the code associated with subspaces in the geometry defined by the lines of an incidence system with a given polarity. If* X *is a partial linear space, then for* $p \in \mathcal{O}$, p^{τ} *is the only element of* $\mathcal{S}$ *incident with* p*; and dually. Examples of polarities of interest are of :*

1. *The incidence system* $\mathcal{P}^{r}_{k,n-k-1}(n,q) = (\mathcal{P}_k(n,q), \mathcal{P}_{n-k-1}(n,q), I_r)$, $0 \leq k \leq [n/2] - 1$ *and* $-1 \leq r \leq k$, *where* $\mathcal{P}_s(n,q)$ *is the set of* $s-$*flats in the the projective space* $\mathcal{P}(n,q)$ *of dimension* n *over* $\mathbb{F}_q$ *with a* $k-$*flat* A *defined to be* I_r*- incident with a* $(n-k-1)-$*flat* B *if* $A \cap B$ *is an* $r-$*flat. Any polarity of* $\mathcal{P}(n,q)$ *induces a polarity on each* $\mathcal{P}^{r}_{k,n-k-1}(n,q)$. FOR A GIVEN r, DOES THERE EXIST ANY OTHER POLARITIES? WHAT IS THE RELATION BETWEEN THE POLARITIES OF $\mathcal{P}(n,q)$ AND THOSE OF $P^{r}_{k,n-k-1}(n,q)$.

2. *The other finite lie incidence structures;*

3. *finite projective planes;*

4. *finite symmetric designs;*

5. *the generalized quadrangle* $W(2^{2n+1})$, $n \geq 1$;

6. *the generalized hexagon* $H(3^{2n+1})$, $n \geq 1$;

7. *the metasymplectic spaces;*

8. ANY INTERESTING EXAMPLES IN DUAL POLAR SPACES, PARAPOLAR SPACES, LIE INCIDENCE SYSTEMS,

Remark 5 *A feature of interest in Example 5 (resp. ly Example 6; the case* $n = 8$ *and* $p = 2$ *of Example 1) is that* $\mathcal{O}$ *is an ovoid of* $W(2^{2n+1})$ *(resp. ly of* $H(3^{2n+1})$*; ?? of*

$\Omega(4,2^e)$ -geometry) and that the characteristic functions of the tangent planes to this ovoid is a basis for the $p-$ ary code generated by the hyperplanes of the incidence system. Also, there is a triality associated with the $\Omega(4,2^e)$ -geometry. What is its relation to the incidence system?

Polarities of symmetric designs

$\mathcal{D}$ be a symmetric $2-(v,k,\lambda)$ design.

For distinct points x,y, the line xy is the intersection of all blocks contains x and y. Two points are on exactly one line and a line of size $(v-\lambda)/(k-\lambda)$ has nonempty intersection with each block [Dembowski and Wagner, Arch. Math. 11 (1960) 465-469].

Assume that $\mathcal{D}$ admits a null polarity τ; that is, $x \in x^\tau$ for each point x. We say that a line is singular if it contains distinct points x,y with $y \in x^\tau$; and nonsingular otherwise.

If xy is a singular line and $x \neq z \in xy$, then $z \in x^\perp \cap y^\perp$, and so, $x,z \subseteq z^\perp$, $y \in xy = xz \subseteq z^\perp$. The equality $xy = xz$ holds because: every block containing x and y contains z also and so, the blocks containing $\{x,y\}$ and $\{x,z\}$ are the same.

Any line in $\mathcal{D}$ is either singular or nonsingular.

Examples: *(1) Projective geometries $\mathcal{PG}(d,q)$*

(2) Highman's orthogonal symmetric designs having the same parameters as in (1): Points are the singular points of a d-dimensional orthogonal $\mathbb{F}_q$-space with both d and q odd; its blocks correspond to the hyperplanes $x^\perp$.

1. (Dembowski-Wagner) The projective space is the only symmetric design such that all lines have size $(v-\lambda)/(k-\lambda)$.

2. (Kantor) Let $\mathcal{D}$ be a symmetric design admitting null polarity.

(a) If all singular lines have size $(v-\lambda)/(k-\lambda) > \lambda$, then $\mathcal{D}$ is either a projective space or an orthogonal design.

(b) If all nonsingular lines have size $(v-\lambda)/(k-\lambda)$, then $\mathcal{D}$ is a projective space.

Survey all known examples

1. Which projective planes admit polarities?

Desarguesian planes do and the polarities are either orthogonal or unitary.

2. For Desarguesian spaces, polarities exist and they are either orthogonal, alternating or unitary.

3. For a polarity of a Desarguesian projective plane, the set of absolute points is nonempty. It has $q+1$ points if the polarity is orthogonal and s^3+1 points if the polarity is unitary and $q=s^2$.

$$Stab_{PGL_3(q)}(\text{ Conic }) = PGL_2(q) = \{y \mapsto (ay+b)/cy+d : ad-bc \neq 0\}.$$

Theorem 6 *Let Γ be a projective plane of order q and θ be a polarity of Γ. Then, the number N_θ of absolute points with respect to θ satisfy $q+1 \leq N_\theta \leq q\sqrt{q}+1$.*

(i) If $N_\theta = q+1$, the set of absolute points is an oval if q is odd and the absolute points is collinear if q is even.

(ii) If $N_\theta = 1 + q\sqrt{q}$, the set of absolute points, with sets of points incident with secant lines, form a unital.

Also, if q is not a square, then $N_\theta = q + 1$.

Proof. ... $\blacksquare$

Definition 7 *A unital of order q is a set of q^3+1 points having $q^2\left(q^3+1\right)/2$ distinguished subsets containing $q+1$ elements each, called blocks, such that each block is determined by any 2 of its points.*

Question: Determine the structure of the subcode of $\mathbb{F}_q$-code of lines of a projective plane π of order q generated by conics of π if q odd and hyperovals of π if q is even.

Problem 8 *For $l \in \mathcal{L}$, can we say anything in general about $|l \cap \mathcal{O}|$?*

Some additional articles:

1. *R. Gow,Construction of exterior powers in Characteristic 2 and spin modules, Geom. Dedicata, 64 (1970), 283- 295*
2. *R.H. Dye : Interrelations of symplectic and orthogonal groups in characteristic two, J. Algebra 59 (1979) pp.202-221.*
3. *R. H. Dye : On the Arf invariant, J. Algebra 53 (1978) 36-39.*
4. *N.F. Inglis : The embedding $O(2m, 2^R) \leq PSp(2m, 2^R)$ Arch. Math. Vol.54 (1990) 327-330.*
5. *J. Diendonne, La geometrie des groupes classique Berlin-Heidelberg-New York, 1955.*
6. *G. Lusztig, Irreducible representations of finite classical groups, Invent. Math. 43 (1977), 125-175*
7. *G. Malle, J. of Algebra 139 (1991) 52-69.*
8. *G. Mason, Varieties attached to an $SL_2\left(2^t\right)$-modules, Proc. of AMS, 111 (1992) 343-350.*
9. *J.L. Alperin and G. Mason, On simple modules for $SL\left(2,q\right)$, Bull. LMS 25 (1993) 17-22.*
10. *P. Sin, The cohomology in degree one of the group F_4 in characteristics 2 with coefficients in a simple module.*
11. *G. Malle, Die unipotenten charakters von ${}^2F_4\left(q^2\right)$, Comm. in Algebra 18 (1990) 236-238.*
12. *G. Malle, Green functions for groups of type E_6 and F_4 in characteristic 2, Comm. in Algebra 21 (1993) 747-798.*

13. T. Shoji, On the green polynomials of Chevalley groups of type F_4, Comm. in Algebra 10 (1982) 505-543.

14. R.M. Marcels and K. Shinoda, Values of the unipotent characters of the Chevalley group of type F_4 at unipotent elements, Tokyo J. Math. 18 (1995) 303-344

15. Cardinali, I.; Lunardon, G.; Polverino, O.; Trombetti, R. Spreads in $H(q)$ and 1-systems of $Q(6,q)$. European J. Combin. 23 (2002), no. 4, 367–376.

16. Bader, Laura; Lunardon, Guglielmo Generalized hexagons and polar spaces. Combinatorics (Assisi, 1996). Discrete Math. 208/209 (1999), 13–22

17. Law, Maska; Penttila, Tim Construction of BLT-sets over small fields. European J. Combin. 25 (2004), no. 1, 1–22.

18. Payne, S. E. Collineations of the generalized quadrangles associated with q-clans. Combinatorics '90 (Gaeta, 1990), 449–461, Ann. Discrete Math., 52, North-Holland, Amsterdam, 1992.

1. Buildings associated with $F_4, E_6, E_7, E_8, ^2F_4$ explicitly and of known metasymplectic spaces.

2. Several realizations of F_4 (As automorphisms of Jordan algebras of exceptional type; as Chevalley groups; as the automorphism group of the Cayley plane).

3. Maximal subgroups of X and X', $X \in \{B_2(q), G_2(q), F_4(q)\}$ in terms of the associated geometry.

4. Does G admit a permutation representation such that, for an algebraic number field K, $A = K[X_1, \cdots, X_n]^G$ is rational; that is, there exist $f_1, \cdots, f_t \in A$ such that $A = K[f_1, \cdots, f_t]$?

5. Ovoids in metasymplectic spaces.

6. Conjugacy classes of elements of G and their identifications with geometric structures.

7. Geometries of type F_4; local characterizations of buildings among geometries of type F_4 and metasymplectic spaces; Tits characterization of buildings of type F_4 among geometries of type F_4.

8. Hirschfeld's approach to Hamada's conjecture applied to codes associated with metasymplectic spaces.

9. Mason varieties for $Sz(2^k), PSp(4, 2^k), F_4(2^k), {}^2F_4(2^k)$-modules; Carleson varieties and Kazhdan-Lusztig varieties.

*10. Generalized Fitting submodules of the modules_*